

A relativistic algorithm with isotropic coordinates

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Abstract We study spherically symmetric spacetimes for matter distributions with isotropic pressures. We generate new exact solutions to the Einstein field equations which also contains isotropic pressures. We develop an algorithm that produces a new solution if a particular solution is known. The algorithm leads to a nonlinear Bernoulli equation which can be integrated in terms of arbitrary functions. We use a conformally flat metric to show that the integrals may be expressed in terms of elementary functions. It is important to note that we utilise isotropic coordinates unlike other treatments.

1 Introduction

We consider the interior of static perfect fluid spheres in general relativity with isotropic pressures. The predictions of general relativity have been shown to be consistent with observational data in relativistic astrophysics and cosmology. For a discussion of the physical features of a gravitating model we require an exact solution to the Einstein field equations. Exact solutions are crucial in the description of dense relativistic astrophysical problems. Many solutions have been found in the past. For some comprehensive lists of known solutions to the field equations refer to Delgaty and Lake [1], Finch and Skea [2], Stephani et al. [3]. Many of these solutions are not physically reasonable. For physical reasonableness we require that the gravitational potentials and matter variables are regular, and well behaved, causality of the spacetime manifold is maintained and values for physical quantities, *e.g.*, the mass of a dense star, are consistent with observation.

Solutions have been found in the past by making assumptions on the gravitational

potentials, matter distribution or imposing an equation of state. These particular approaches do yield models which have interesting properties. However in principle it would be desirable to have a general method that produces exact solutions in a systematic manner. Some systematic methods generated in the past are those of Rahman and Visser [4], Lake [5], Martin and Visser [6], Boonserm et al. [7], Herrera et al. [8], Chaisi and Maharaj [9] and Maharaj and Chaisi [10]. In general relativity we have the freedom of using any well defined coordinate system. The references mentioned above mainly use canonical coordinates. The use of isotropic coordinates may provide new insights and possibly lead to new solutions. This is the approach that we follow in this paper. We generate a new algorithm producing a new solution to Einstein field equations in isotropic coordinates. From a given solution we can find a new solution with isotropic pressures.

The object of this paper is to find new classes of exact solutions of the Einstein field equations with an uncharged isotropic matter distribution from a given seed metric. In Section 2, we derive the Einstein field equations for neutral perfect fluids in static spherically symmetric spacetime. We introduce new variables due to Kustaanheimo and Qvist [11] to rewrite the field equations and the condition of pressure isotropy in equivalent forms. In Section 3, we introduce our algorithm and the master nonlinear second order differential equation containing two arbitrary functions, that has to be solved. In Section 4, we present new classes of exact solutions in terms of the arbitrary functions. In Section 5, we give an example for a conformally flat metric showing that the integrals generated in Section 4 may be explicitly evaluated. In Section 6, we summarise the results obtained in this paper.

2 The model

We are modelling the interior of a dense relativistic star in strong gravitational fields. The line element of the interior spacetime, with isotropic coordinates, has the following form

$$ds^2 = -A^2(r)dt^2 + B^2(r)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \quad (1)$$

where $A(r)$ and $B(r)$ are arbitrary functions representing the gravitational potentials. Relativistic compact objects such as neutron stars in astrophysics are described by this line element. The energy momentum tensor for the interior of the star has the form of a perfect fluid

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab} \quad (2)$$

where ρ is the energy density and p is the isotropic pressure. These quantities are measured relative to a timelike unit four-velocity u^a ($u^a u_a = -1$).

The Einstein field equations for (1) and (2) have the form

$$\rho = -\frac{1}{B^2} \left[2\frac{B''}{B} - \frac{B'}{B} \left(\frac{B'}{B} - \frac{4}{r} \right) \right] \quad (3a)$$

$$p = 2\frac{A'}{A} \left(\frac{B'}{B^3} + \frac{1}{r} \frac{1}{B^2} \right) + \frac{B'}{B^3} \left(\frac{B'}{B} + \frac{2}{r} \right) \quad (3b)$$

$$p = \frac{1}{B^2} \left(\frac{A''}{A} + \frac{1}{r} \frac{A'}{A} \right) + \frac{1}{B^2} \left[\frac{B''}{B} - \frac{B'}{B} \left(\frac{B'}{B} - \frac{1}{r} \right) \right] \quad (3c)$$

in isotropic coordinates. Primes denote differentiation with respect to the radial coordinate r . On equating (3b) and (3c) we obtain the condition of pressure isotropy which has the form:

$$\frac{A''}{A} + \frac{B''}{B} = \left(\frac{A'}{A} + \frac{B'}{B} \right) \left(2\frac{B'}{B} + \frac{1}{r} \right) \quad (4)$$

This is the master equation which has to be integrated to produce an exact solution to the field equations.

It is possible to write the system (3) in an equivalent form by introducing new variables. We utilize a transformation that has proven to be helpful in relativistic stellar physics. We introduce the new variables

$$x \equiv r^2, \quad L \equiv B^{-1}, \quad G \equiv LA \quad (5)$$

The above transformation was first suggested by Kustaanheimo and Qvist [11]. On applying transformation (5) in the field equations (3) we obtain the equivalent system

$$\rho = 4[2xLL_{xx} - 3(xL_x - L)L_x] \quad (6a)$$

$$p = 4L(L - 2xL_x)\frac{G_x}{G} - 4(2L - 3xL_x)L_x \quad (6b)$$

$$p = 4xL^2\frac{G_{xx}}{G} + 4L(L - 2xL_x)\frac{G_x}{G} - 4(2L - 3xL_x)L_x - 8xLL_{xx} \quad (6c)$$

We note that the above equations (6) are highly nonlinear in both L and G . In this system there are three independent equations and four unknowns ρ , p , L and G . So

we need to choose the functional form for L or G in order to integrate and obtain an exact solution. The value of the transformation (5) is highlighted in the reduction of the condition of pressure isotropy. On equating equations (6b) and (6c) we get

$$LG_{xx} = 2GL_{xx} \quad (7)$$

which is the new condition of pressure isotropy which has a simpler compact form.

3 The algorithm

It is possible to find new solutions to the Einstein's equations from a given seed metric. Examples of this process are given in the treatments of Chaisi and Maharaj [9] and Maharaj and Chaisi [10]. They found new models, with anisotropic pressures, from a given seed isotropic metric in Schwarzschild coordinates. Our intention is to find new models, with isotropic pressures, from a given solution in terms of the isotropic line element (1).

We can provide some new classes of exact solutions to the Einstein field equations by generating a new algorithm that produces a model from a given solution. We assume a known solution of the form (\bar{L}, \bar{G}) so that

$$\bar{L}\bar{G}_{xx} = 2\bar{G}\bar{L}_{xx} \quad (8)$$

holds. We seek a new solution (L, G) given by

$$L = \bar{L}e^{g(x)}, \quad G = \bar{G}e^{f(x)} \quad (9)$$

where $f(x)$ and $g(x)$ are arbitrary functions. On substituting equation (9) into (7) we obtain

$$(\bar{L}\bar{G}_{xx} - 2\bar{G}\bar{L}_{xx}) + 2(\bar{L}\bar{G}_x f_x - 2\bar{G}\bar{L}_x g_x) + \bar{L}\bar{G}(f_{xx} - 2g_{xx}) + \bar{L}\bar{G}(f_x^2 - 2g_x^2) = 0 \quad (10)$$

which is given in terms of two arbitrary functions $f(x)$ and $g(x)$. Then realizing that (\bar{L}, \bar{G}) is a solution of (7) and using (8) we obtain the reduced result

$$(f_{xx} - 2g_{xx}) + 2\left(\frac{\bar{G}_x}{\bar{G}}f_x - 2\frac{\bar{L}_x}{\bar{L}}g_x\right) + (f_x^2 - 2g_x^2) = 0 \quad (11)$$

We need to demonstrate the existence of functions $f(x)$ and $g(x)$ that satisfy (11). In general it is difficult to integrate equation (11), since it is given in terms of two arbitrary functions which are nonlinear.

4 New solutions

We consider several cases of equation (11) for which we have been able to complete the integration.

4.1 $g(x)$ is specified

We can integrate (11) if $g(x)$ is specified. As a simple example we take $g(x) = 1$. Then (11) becomes

$$f_{xx} + 2\frac{\bar{G}_x}{\bar{G}}f_x + f_x^2 = 0 \quad (12)$$

which is nonlinear in f . This is a first order Bernoulli equation in f_x . We can rewrite (12) in the form

$$\left(\frac{1}{f_x}\right)_x - 2\left(\frac{\bar{G}_x}{\bar{G}}\right)\left(\frac{1}{f_x}\right) = 1 \quad (13)$$

It is possible to integrate equation (13) since it is linear in $\frac{1}{f_x}$ to obtain

$$f_x = \bar{G}^{-2} \left(\int \bar{G}^{-2} dx + c_1 \right)^{-1} \quad (14)$$

We can formally integrate (14) to obtain the function $f(x)$ as

$$f(x) = \int \left[\bar{G}^{-2} \left(\int \bar{G}^{-2} dx + c_1 \right)^{-1} \right] dx + c_2 \quad (15)$$

where c_1 and c_2 are arbitrary constants.

Then the new solution to equation (7) has the form

$$L = \bar{L} \quad (16a)$$

$$G = \bar{G} \exp \left(\int \left[\bar{G}^{-2} \left(\int \bar{G}^{-2} dx + c_1 \right)^{-1} \right] dx + c_2 \right) \quad (16b)$$

Therefore we have shown that if a solution (\bar{L}, \bar{G}) to the field equations is known, then a new solution (L, G) is given by (16).

4.2 $f(x)$ is specified

We can also integrate (11) if $f(x)$ is specified. As another simple example we take $f(x) = 1$. Then equation (11) becomes

$$g_{xx} + 2\frac{\bar{L}_x}{\bar{L}}g_x + g_x^2 = 0 \quad (17)$$

which is nonlinear in g . This is a first order Bernoulli equation in g_x . The differential equation (17) has a form similar to (12) in section 4.1. Following the same procedure we obtain

$$g(x) = \int \left[\bar{L}^{-2} \left(\int \bar{L}^{-2} dx + c_1 \right)^{-1} \right] dx + c_2 \quad (18)$$

where c_1 and c_2 are arbitrary constants.

Then another new solution to equation (7) is given by

$$G = \bar{G} \quad (19a)$$

$$L = \bar{L} \exp \left(\int \left[\bar{L}^{-2} \left(\int \bar{L}^{-2} dx + c_1 \right)^{-1} \right] dx + c_2 \right) \quad (19b)$$

Therefore we have determined that if a solution (\bar{L}, \bar{G}) to the field equations is known then a new solution (L, G) is given by (19). Note that the solution (19) is different from that presented in (16).

4.3 $g(x) = \alpha f(x)$

We can integrate (11) if a relationship between the functions $f(x)$ and $g(x)$ exists. We illustrate this feature by assuming that

$$g(x) = \alpha f(x) \quad (20)$$

where α is an arbitrary constant. Then (11) becomes

$$f_{xx} + \frac{2}{1-2\alpha} \left(\frac{\bar{G}_x}{\bar{G}} - 2\alpha \frac{\bar{L}_x}{\bar{L}} \right) f_x + \left(\frac{1-2\alpha^2}{1-2\alpha} \right) f_x^2 = 0 \quad (21)$$

which is a first order Bernoulli equation in f_x . For convenience we let

$$\Theta = \left(\frac{1-2\alpha^2}{1-2\alpha} \right), \quad \eta = \frac{2}{1-2\alpha}, \quad \alpha \neq \frac{1}{2} \quad (22)$$

so that we can write (21) as

$$\left(\frac{1}{f_x} \right)_x - \eta \left(\frac{\bar{G}_x}{\bar{G}} - 2\alpha \frac{\bar{L}_x}{\bar{L}} \right) \left(\frac{1}{f_x} \right) = \Theta \quad (23)$$

which is linear in $\frac{1}{f_x}$. We integrate (23) to obtain

$$f_x = \left(\frac{\bar{L}^{2\alpha}}{\bar{G}} \right)^\eta \left[\Theta \int \left(\frac{\bar{L}^{2\alpha}}{\bar{G}} \right)^\eta dx + c_1 \right]^{-1} \quad (24)$$

We now formally integrate (24) to obtain

$$f(x) = \int \left(\left(\frac{\bar{L}^{2\alpha}}{\bar{G}} \right)^\eta \left[\Theta \int \left(\frac{\bar{L}^{2\alpha}}{\bar{G}} \right)^\eta dx + c_1 \right]^{-1} \right) dx + c_2 \quad (25)$$

where c_1 and c_2 are constants.

We now have a new solution of (7) given by

$$L = \bar{L} \exp \alpha \left[\int \left(\left(\frac{\bar{L}^{2\alpha}}{\bar{G}} \right)^\eta \left[\Theta \int \left(\frac{\bar{L}^{2\alpha}}{\bar{G}} \right)^\eta dx + c_1 \right]^{-1} \right) dx + c_2 \right] \quad (26a)$$

$$G = \bar{G} \exp \left[\int \left(\left(\frac{\bar{L}^{2\alpha}}{\bar{G}} \right)^\eta \left[\Theta \int \left(\frac{\bar{L}^{2\alpha}}{\bar{G}} \right)^\eta dx + c_1 \right]^{-1} \right) dx + c_2 \right] \quad (26b)$$

where Θ and η are given in (22). Therefore we have demonstrated that if a solution (\bar{L}, \bar{G}) to the field equations is specified then a new solution (L, G) is provided by (26).

Some special cases related to (26) should be pointed out. These relate to $\alpha = 1$, $\pm \frac{1}{\sqrt{2}}$, $\frac{1}{2}$. We consider each in turn.

Case (i): $\alpha = 1$

With $\alpha = 1$ we find that (26) becomes

$$L = \bar{L} \exp \left(\int \left[\frac{\bar{G}^2}{\bar{L}^4} \left(\int \frac{\bar{G}^2}{\bar{L}^4} dx + c_1 \right)^{-1} \right] dx + c_2 \right) \quad (27a)$$

$$G = \bar{G} \exp \left(\int \left[\frac{\bar{G}^2}{\bar{L}^4} \left(\int \frac{\bar{G}^2}{\bar{L}^4} dx + c_1 \right)^{-1} \right] dx + c_2 \right) \quad (27b)$$

which is a simple form.

Case (ii): $\alpha = \pm \frac{1}{\sqrt{2}}$

If we set $\alpha = \pm \frac{1}{\sqrt{2}}$ then (26) becomes

$$L = \bar{L} \exp \left[\pm \frac{1}{\sqrt{2}} \left(c_1 \int \left(\frac{\bar{L}^{\pm\sqrt{2}}}{\bar{G}} \right)^{\frac{2}{1-(\pm\sqrt{2})}} dx + c_2 \right) \right] \quad (28a)$$

$$G = \bar{G} \exp \left[c_1 \int \left(\frac{\bar{L}^{\pm\sqrt{2}}}{\bar{G}} \right)^{\frac{2}{1-(\pm\sqrt{2})}} dx + c_2 \right] \quad (28b)$$

which is another simple case.

Case (iii): $\alpha = \frac{1}{2}$

If $\alpha = \frac{1}{2}$ then (26) is not valid. For this case, equation (11) becomes

$$f_x \left[f_x + 4 \left(\frac{\bar{G}_x}{\bar{G}} - \frac{\bar{L}_x}{\bar{L}} \right) \right] = 0 \quad (29)$$

When f is constant then g is also constant by (20); then (7) does not produce a new solution because of (9). When f is not constant then we can integrate (29) to produce the solution

$$L = K \frac{\bar{L}^3}{\bar{G}^2} \quad (30a)$$

$$G = K \frac{\bar{L}^4}{\bar{G}^3} \quad (30b)$$

where K is a constant. Thus $\alpha = \frac{1}{2}$ generates another new solution (L, G) to (11).

5 Example

We show by means of a specific example that the integrals generated in section 4 may be evaluated to produce a new exact solution to the field equations in terms of elementary functions. In our example we choose

$$\bar{L} = b + ax \quad (31a)$$

$$\bar{G} = 1 + cx \quad (31b)$$

Then the corresponding line element is given by

$$ds^2 = - \left(\frac{1 + cr^2}{b + ar^2} \right)^2 dt^2 + \left(\frac{1}{b + ar^2} \right)^2 (dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)) \quad (32)$$

which is conformally flat. The energy density for the metric (32) is constant so that we have the Schwarzschild interior solution in isotropic coordinates.

Conformally flat metrics are important in gravitational physics in a general relativistic setting. They arise, for instance in the gravitational collapse of a radiating star, as shown in the treatments of Herrera et al. [12], Maharaj and Govender [13], Mistry et al. [14] and Abebe et al. [15]. For the choice of (31) we find that (27) becomes

$$L = (b + ax) \exp \left(\int \left[\frac{(1 + cx)^2}{(b + ax)^4} \left(\int \frac{(1 + cx)^2}{(b + ax)^4} dx + c_1 \right)^{-1} \right] dx + c_2 \right) \quad (33a)$$

$$G = (1 + cx) \exp \left(\int \left[\frac{(1 + cx)^2}{(b + ax)^4} \left(\int \frac{(1 + cx)^2}{(b + ax)^4} dx + c_1 \right)^{-1} \right] dx + c_2 \right) \quad (33b)$$

The integrals in (33) can be evaluated and we obtain

$$L = \frac{1}{(b + ax)^2} U(x) \quad (34a)$$

$$G = \frac{(1 + cx)}{(b + ax)^3} U(x) \quad (34b)$$

where $c_1 = 0$ and $c_2 = 1$ and we have set

$$U(x) = b^2 c^2 + abc(1 + 3cx) + a^2(1 + 3cx + 3c^2 x^2) \quad (35)$$

Thus the known solution (\bar{L}, \bar{G}) in (31) produces a new solution (L, G) in (34). The line element for the new solution has the form

$$ds^2 = - \left(\frac{1 + cr^2}{b + ar^2} \right)^2 dt^2 + \left(\frac{(b + ar^2)^2}{U(r)} \right)^2 (dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)) \quad (36)$$

where $U(r)$ is given by (35). Thus our algorithm has produced a new (not conformally flat) solution to the Einstein's field equations from a seed conformally flat model.

6 Conclusion

We now comment on the physical properties of the example. We have generated plots for the energy density ρ , pressure p , and the speed of sound in Figures 1-3, respectively.

These graphical plots indicate that ρ and p are positive and well behaved. The speed of sound is less than the speed of light as required for causality. Therefore the algorithm presented in this paper produces new solutions which are physically reasonable.

We have generated an algorithm to produce a new solution to the Einstein field equations from a given seed metric. We observe that the resulting model contains isotropic pressures unlike the approach of Chaisi and Maharaj [9] and Maharaj and Chaisi [10]; in their treatment the new model has anisotropic pressures. Another advantage of our approach is the use of isotropic coordinates in the formulation of the condition of pressure isotropy. This may lead to new insights into the behaviour of gravity since previous treatments mainly utilised canonical coordinates. The algorithm produced a new solution in terms of integrals containing arbitrary functions. We have shown, with the help of a conformally flat metric, that these integrals may be evaluated in terms of elementary functions. This example suggests that our approach may be extended to other physically relevant metrics.

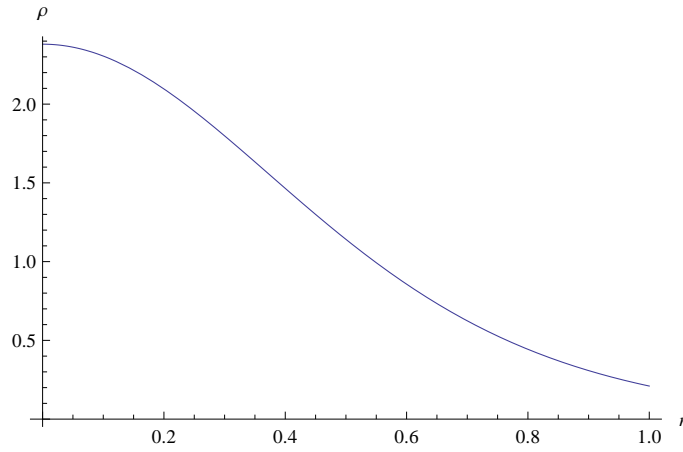


Figure 1: Energy density ρ

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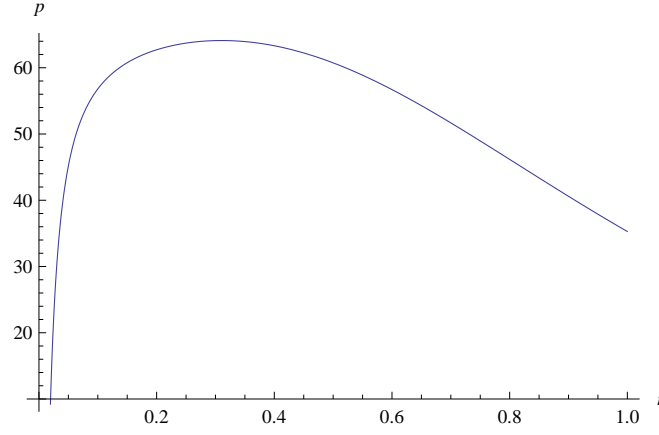


Figure 2: Pressure p

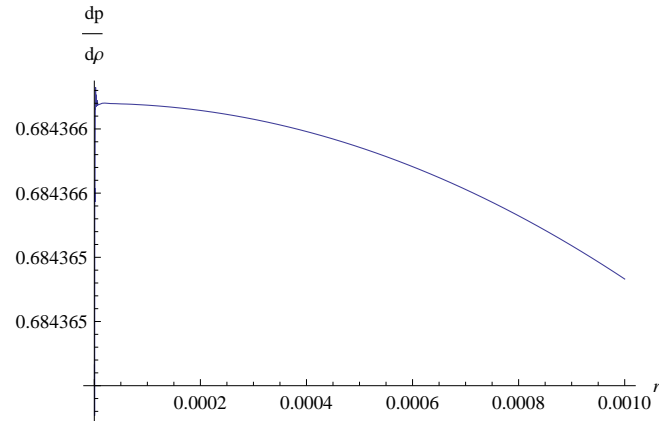


Figure 3: Speed of sound $\frac{dp}{d\rho}$

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